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Carlson's Theorem for Harmonic Functions

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Carlson's theorem [1, p. 153] states that an entire function of exponential type less than π must vanish identically if it vanishes at the integers. The problem of extending Carlson's theorem to harmonic functions $u(z)$ was posed in [2], where Boas showed that $u(z) \equiv 0$ provided it vanishes on two parallel lines of lattice points.

Let $f(z) = u(z) + iv(z)$, with $u(m) = 0$, $-\infty < m < \infty$. By assuming $\{f(m)\} \in l^1$, it can be shown that $f(z) \equiv 0$ provided that $u(m+i) = 0$ or $v(m+i) = 0$, $-\infty < m < \infty$ (see [5, pp. 3–9]). In this paper I shall prove that $u(z)$ need only vanish at $z = m + i$, $-\infty < m < \infty$, in order to vanish identically. The proof relies heavily on the assumption that $\{f(m)\} \in l^1$, for then $f(z)$ has the representation

$$f(z) = f'(0)\frac{\sin \pi z}{\pi} + f(0)\frac{\sin \pi z}{\pi z} + \frac{z \sin \pi z}{\pi} \sum_{n \neq 0} \frac{(-1)^n f(n)}{n(z-n)}$$

(see [1, p. 221]), and $u(m+i)$ is the mth Fourier coefficient of

$$F(x) = \cosh x \sum_{-\infty}^{\infty} u(n)\, e^{inx} + i \sinh x \sum_{-\infty}^{\infty} v(n)\, e^{inx}, \qquad |x| < \pi$$

(see [5, pp. 17–18]). By considering $g(z) = \overline{f(\bar{z} + i)}$, we may restate the main result as follows.

THEOREM. *Let $f(z)$ be an entire function of exponential type less than π, with $\{f(m+i)\} \in l^1$. If $\operatorname{Re} f(m) = 0$, $-\infty < m < \infty$, then $f(z) \equiv 0$.*

Hence a harmonic function must vanish identically if it vanishes at the integers and belongs to l^1 on a parallel line of lattice points. Note the necessity of the growth restriction: consider the real part of $f(z) = iz$.

Let us begin by establishing the following two lemmas.

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LEMMA 1. *Let $f(z)$ be an entire function of exponential type τ less than π. Let $f(z)=u(z)+iv(z)$ (u, v, real), $\{f(m)\}\in l^1$, and $u(m+i)=0$, $-\infty<m<\infty$. If $v(m)=0$, $m=0, \pm 1, \pm 2, \ldots$, then $f(z)\equiv 0$.*

Proof. Since $f(z)$ is of exponential type τ less than π and is bounded at the integers, it follows from Cartwright's theorem [1, p. 180] that $f(x)$ is bounded for all real x. Hence $f(z)$ has the representation

$$f(z)=f'(0)\frac{\sin \pi z}{\pi}+f(0)\frac{\sin \pi z}{\pi z}+\frac{z \sin \pi z}{\pi}\sum_{n\neq 0}\frac{(-1)^n f(n)}{n(z-n)}$$

[1, p. 221]. Set $z=m+i$, keeping in mind that $f(n)=u(n)$ and $\sin \pi(m+i)=i(-1)^m \sinh \pi$, to obtain

$$\begin{aligned} f(m+i)=&\, v'(0)(-1)^{m+1}\frac{\sinh \pi}{\pi}+iu'(0)(-1)^m\frac{\sinh \pi}{\pi}\\ &+iu(0)\frac{\sinh \pi}{\pi}\left[\frac{(-1)^m m}{m^2+1}\right]+u(0)\frac{\sinh \pi}{\pi}\left[\frac{(-1)^m}{m^2+1}\right]\\ &+im(-1)^m\frac{\sinh \pi}{\pi}\sum_{n\neq 0}\frac{(-1)^n u(n)}{n(m-n+i)}\\ &-(-1)^m\frac{\sinh \pi}{\pi}\sum_{n\neq 0}\frac{(-1)^n u(n)}{n(m-n+i)}. \end{aligned}$$

Multiply both summands on the right of the above equation by $((m-n)-i)/((m-n)-i)$ to obtain

$$\begin{aligned} f(m+i)=&\, v'(0)(-1)^{m+1}\frac{\sinh \pi}{\pi}+iu'(0)(-1)^m\frac{\sinh \pi}{\pi}\\ &+iu(0)\frac{\sinh \pi}{\pi}\left[\frac{(-1)^m m}{m^2+1}\right]+u(0)\frac{\sinh \pi}{\pi}\left[\frac{(-1)^m}{m^2+1}\right]\\ &+im\frac{\sinh \pi}{\pi}\sum_{n\neq 0}\frac{(-1)^{m-n}(m-n)\,u(n)}{n[(m-n)^2+1]}\\ &+m\frac{\sinh \pi}{\pi}\sum_{n\neq 0}\frac{(-1)^{m-n}u(n)}{n[(m-n)^2+1]}\\ &-\frac{\sinh \pi}{\pi}\sum_{n\neq 0}\frac{(-1)^{m-n}(m-n)\,u(n)}{n[(m-n)^2+1]}\\ &+i\frac{\sinh \pi}{\pi}\sum_{n\neq 0}\frac{(-1)^{m-n}u(n)}{n[(m-n)^2+1]}. \end{aligned}$$

We then have

$$\operatorname{Re} f(m+i) = u(m+i) = \frac{\sinh \pi}{\pi}\left[v'(0)(-1)^{m+1} + u(0)\frac{(-1)^m}{m^2+1} \right.$$
$$\left. + m \sum_{n \neq 0} \frac{(-1)^{m-n} u(n)}{n[(m-n)^2+1]} - \sum_{n \neq 0} \frac{(-1)^{m-n}(m-n)\, u(n)}{n[(m-n)^2+1]} \right],$$
$$\operatorname{Im} f(m+i) = v(m+i) = \frac{\sinh \pi}{\pi}\left[u'(0)(-1)^{m} + u(0)\frac{(-1)^m m}{m^2+1} \right.$$
$$\left. + m \sum_{n \neq 0} \frac{(-1)^{m-n}(m-n)\, u(n)}{m[(m-n)^2+1]} + \sum_{n \neq 0} \frac{(-1)^{m-n} u(n)}{n[(m-n)^2+1]} \right].$$

By [4, p. 12] with $a=b=1$, $l=\pi$,

$$\frac{(-1)^m}{m^2+1}$$

is the mth Fourier coefficient of

$$\frac{\pi \cosh x}{\sinh \pi}, \qquad -\pi \leqslant x \leqslant \pi.$$

Hence,

$$u(0)\frac{\sinh \pi}{\pi}\left[\frac{(-1)^m}{m^2+1}\right]$$

is the mth Fourier coefficient of

$$u(0) \cosh x, \qquad -\pi \leqslant x \leqslant \pi.$$

Since the convolution of the Fourier coefficients of two functions yields the Fourier coefficients of the product of the functions [3, p. 23], we have that

$$\frac{\sinh \pi}{\pi}\left[m \sum_{n \neq 0} \frac{(-1)^{m-n} u(n)}{n[(m-n)^2+1]} \right]$$

is the mth Fourier coefficient of

$$\frac{1}{i}\frac{d}{dx}\left[\cosh x \sum_{n \neq 0} \frac{u(n)}{n} e^{inx} \right], \qquad -\pi \leqslant x \leqslant \pi.$$

Similarly, by [4, p. 13], we have, for $a=b=1$, $l=\pi$,

$$\frac{(-1)^m m}{m^2+1}$$

is the mth Fourier coefficient of

$$\frac{-i\pi \sinh x}{\sinh \pi}, \qquad -\pi < x < \pi,$$

so that

$$\frac{-\sinh \pi}{\pi}\left[\sum_{n \neq 0} \frac{(-1)^{m-n}(m-n)\, u(n)}{n[(m-n)^2+1]}\right]$$

is the mth Fourier coefficient of

$$i \sinh x \sum_{n \neq 0} \frac{u(n)}{n} e^{inx}, \qquad -\pi < x < \pi.$$

Assuming $v'(0) = 0$, we have

$$u(m+i) = \frac{\sinh \pi}{\pi}\left[u(0)\frac{(-1)^m}{m^2+1} + m \sum_{n \neq 0} \frac{(-1)^{m-n}\, u(n)}{n[(m-n)^2+1]} - \sum_{n \neq 0} \frac{(-1)^{m-n}(m-n)\, u(n)}{n[(m-n)^2+1]}\right]$$

is the mth Fourier coefficient of

$$F(x) = \frac{\sinh \pi}{\pi}\left[\frac{u(0)\, \pi \cosh x}{\sinh \pi} + \frac{1}{i}\frac{d}{dx}\left\{\frac{\pi \cosh x}{\sinh \pi} \sum_{n \neq 0} \frac{u(n)}{n} e^{inx}\right\} + \frac{i\pi \sinh x}{\sinh \pi} \sum_{n \neq 0} \frac{u(n)}{n} e^{inx}\right], \qquad -\pi < x < \pi.$$

Simplifying, we find that $u(m+i)$ is the mth Fourier coefficient of

$$F(x) = u(0) \cosh x - i \sinh x \sum_{n \neq 0} \frac{u(n)}{n} e^{inx} + \cosh x \sum_{n \neq 0} u(n)\, e^{inx} + i \sinh x \sum_{n \neq 0} \frac{u(n)}{n} e^{inx},$$

$$F(x) = \cosh x \sum_{-\infty}^{\infty} u(n)\, e^{inx}, \qquad -\pi < x < \pi.$$

The hypothesis implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\, e^{-imx}\, dx = 0, \qquad -\infty < m < \infty.$$

By the completeness of $\{e^{imx}\}$ $(-\infty < m < \infty)$, over $(-\pi, \pi)$ we have that $F(x)$ is equivalent to zero. Since $\cosh x$ never vanishes, we conclude that $u(n) \equiv 0$. Hence $f(n) = iv(n) = 0$ for all n, and by Carlson's theorem [1, p. 153] $f(z) \equiv 0$.

If $v'(0)$ does not vanish, then $u(m+i) + ((\sinh \pi)/\pi)\, v'(0)(-1)^m$ is the mth Fourier coefficient of $F(x)$. By hypothesis, $u(m+i) = 0$ for all m. Since the Fourier coefficients of $F(x)$ must approach zero, we must have $v'(0) = 0$. Now proceed as above and conclude that $f(z)$ is equivalent to zero.

LEMMA 2. *With the hypotheses of Lemma* 1, *if* $v(m+i) = 0$, $-\infty < m < \infty$, *then* $f(z) \equiv 0$.

Proof. Consider $g(z) = -if(z)$.

We now prove the main result.

THEOREM. *Let* $f(z)$ *be an entire function of exponential type* τ *less than* π, *with* $\{f(m+i)\} \in l^1$. *If* $\operatorname{Re} f(m) = 0$, $-\infty < m < \infty$, *then* $f(z) \equiv 0$.

Proof. Let $g(z) = \overline{f(\bar{z}+i)}$. If $g(z) = u(z) + iv(z)$ $(u, v,$ real), then by [5, p. 18] we have that $u(m+i)$ is the mth Fourier coefficient of

$$F(x) = \cosh x \sum_{-\infty}^{\infty} u(n)\, e^{inx} + i \sinh x \sum_{-\infty}^{\infty} v(n)\, e^{inx}, \qquad |x| < \pi. \tag{1}$$

The hypothesis implies that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)\, e^{-imx}\, dx = 0, \qquad -\infty < m < \infty,$$

so that $F(x)$ vanishes identically. Therefore the real and imaginary parts of $F(x)$ must vanish, so that we have

$$\cosh x \sum_{-\infty}^{\infty} u(n) \cos nx - \sinh x \sum_{-\infty}^{\infty} v(n) \sin nx = 0$$

$$\cosh x \sum_{-\infty}^{\infty} u(n) \sin nx + \sinh x \sum_{-\infty}^{\infty} v(n) \cos nx = 0.$$

Written in matrix form, the above equations become

$$\begin{bmatrix} \sum_{-\infty}^{\infty} u(n) \cos nx & -\sum_{-\infty}^{\infty} v(n) \sin nx \\ \sum_{-\infty}^{\infty} u(n) \sin nx & \sum_{-\infty}^{\infty} v(n) \cos nx \end{bmatrix} \begin{bmatrix} \cosh x \\ \sinh x \end{bmatrix} = 0$$

and this system has a nontrivial solution if and only if its determinant vanishes. Hence we have

$$\sum_{-\infty}^{\infty} u(n) \cos nx \sum_{-\infty}^{\infty} v(m) \cos mx + \sum_{-\infty}^{\infty} u(n) \sin nx \sum_{-\infty}^{\infty} v(m) \sin mx = 0,$$

or

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} u(n)\, v(m) \cos(n-m)\, x = 0. \tag{2}$$

Fix k an integer, and multiply (2) by $\cos kx$ and integrate over $(-\pi, \pi)$. We obtain

$$\int_{-\pi}^{\pi} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} u(n)\, v(m) \cos(n-m)\, x \cos kx\, dx = 0. \tag{3}$$

Now the left-hand side of (3) has the value zero whenever $n - m \neq k$. For $n - m = k$, the integral has the value $\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} u(n)\, v(m)$. Hence,

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} u(n)\, v(m) = 0, \qquad n - m = k$$

and

$$\sum_{n=-\infty}^{\infty} u(n)\, v(n-k) = 0, \qquad \text{for all } k. \tag{4}$$

Let $U(x) = \sum_{-\infty}^{\infty} u(n)\, e^{inx}$, $V(x) = \sum_{-\infty}^{\infty} v(m)\, e^{imx}$, $|x| < \pi$. Then (4) implies that

$$U(x)\, V(x) = 0. \tag{5}$$

Squaring (1) and using (5), we have that

$$[F(x)]^2 = \cosh^2 x[U(x)]^2 - \sinh^2 x[V(x)]^2 = 0. \tag{6}$$

Since $F(x) = 0$, (1) implies that

$$\cosh xU(x) = -i \sinh xV(x)$$

and

$$\cosh^2 x[U(x)]^2 = -\sinh^2 x[V(x)]^2. \tag{7}$$

Substitute (7) into (6) and obtain

$$2 \cosh^2 x[U(x)]^2 = 0.$$

Since $\cosh x$ never vanishes, we have $\sum_{-\infty}^{\infty} u(n)\, e^{inx} = 0$ and $u(n) = 0$ for all n. By [5, p. 3], $g(z) \equiv 0$, and so $f(z) \equiv 0$.

When substituting (7) in (6) we may also obtain

$$-2 \sinh^2 x[V(x)]^2 = 0.$$

Hence $\sum_{-\infty}^{\infty} v(n)\, e^{inx} = 0$ and $v(n) = 0$ for all n. By Lemma 1, $g(z) \equiv 0$, and so $f(z) \equiv 0$.

COROLLARY. *Let $f(z)$ be an entire function of exponential type τ less than π, with $\{f(m+i)\} \in l^1$. If* $\operatorname{Im} f(m) = 0$, $-\infty < m < \infty$, *then* $f(z) \equiv 0$.

Proof. Consider $h(z) = -if(z)$. Now proceed as in the proof of the theorem, and use [5, p. 8] and Lemma 2.

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